

The Twin Paradox
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In the following calculation, we are assuming that there are exactly 365.25 days in a year and that the length of a month is exactly 1/12 of a year. The calculator used was capable of keeping track of 15 significant figures.

Albert lives on Earth. Henry is traveling in a spaceship, making a round trip to the planet, Gog, $L = 10$ light years away. Henry's speed during this trip is $v = 0.9999652c$. At this speed, the value of γ is

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}} = 119.866868.$$

In Albert's frame of reference, Henry's trip takes a total time equal to

$$t = \frac{2L}{v} = 20.0006960 \text{ y.}$$

In Albert's frame of reference, Henry's clock is running slow by a factor of γ , so according to Henry's moving clock, the trip takes a total time equal to

$$t' = \frac{t}{\gamma} = 2.0022910 \text{ mo.}$$

In Henry's frame of reference, the distance to Gog is length-contracted by a factor of γ , so he also thinks that the trip takes a total time equal to

$$t' = \frac{2(L/\gamma)}{v} = \frac{t}{\gamma} = 2.0022910 \text{ mo.}$$

During the trip, Albert sends Henry a beep once per second. We can consider this a signal with a frequency equal to $f = 1$ Hz. In Henry's frame of reference, Albert's clock is running slow by a factor of γ , so the frequency of Albert's signal is

$$f_S = \frac{f}{\gamma} = 8.34258886 \times 10^{-3} \text{ Hz.}$$

During the trip to Gog, Henry is traveling away from Albert, so this signal experiences a Doppler shift. Henry observes the frequency of the signal to be

$$f_{O1} = f_S \left(\frac{c}{c+v} \right) = 4.17136701 \times 10^{-3} \text{ Hz.}$$

(This is the same as the Doppler shift of sound due to a moving source.) During the time Henry is traveling from Earth to Gog, he receives $f_{O1}t'/2 = 10982$ beeps from Earth. This is a little over 3 h in Earth's time. He is traveling so fast that the light from Earth has a difficult time catching up with him, and he can only observe about 3 h of Earth's time during his trip to Gog.

On the way back to Earth, Henry is traveling toward Albert, and this signal experiences a different Doppler shift. Henry observes the frequency of the signal to be

$$f_{O2} = f_S \left(\frac{c}{c-v} \right) = 239.729565 \text{ Hz.}$$

During the time Henry is traveling from Gog to Earth, he receives $f_{O2}t'/2 = 631162982$ beeps from Earth. This is about 20 y in Earth's time. The Doppler shift allows him to see 20 y of Earth's time during his 1-month trip back to Earth. Albert's "slow" clock on Earth has been Doppler shifted so that it appears to Henry to be running fast. If Henry had a good telescope, he would see Earth revolve around the sun 20 times during his return trip.

The total number of beeps received by Henry during his round trip is $10982 + 631162982 = 631173964$ which is 20.0006960 y in Earth's time, exactly equal to the total time of the trip in Albert's frame of reference.

During the round trip, Henry also sends Albert beeps once per second. In Albert's frame of reference, Henry's clock is running slow by a factor of γ , so the frequency of Henry's signal is $f_S = 8.34258886 \times 10^{-3}$ Hz, the same as the frequency of Albert's signal in Henry's frame of reference.

During the trip to Gog, Henry is traveling away from Albert, so this signal experiences a Doppler shift. Albert observes a frequency equal to $f_{O1} = 4.17136701 \times 10^{-3}$ Hz, the same frequency Henry observed when he detected beeps coming from Albert. Henry is sending this signal during his entire trip to Gog (about 10 y in Albert's frame of reference). However, the last beep sent by Henry when he reaches Gog requires another 10 y to reach Albert. So Albert receives the signal for a total amount of time equal to $t_1 = t/2 + L/c = (20.0006960 \text{ y})/2 + 10 \text{ y} = 20.0003480 \text{ y}$, and he receives $f_{O1}t_1 = 2632812$ beeps that Henry sent while he was traveling to Gog. This is about 1 mo in Henry's time, which is how long the trip took in Henry's frame of reference.

On the way back to Earth, Henry is traveling toward Albert, and Albert observes a frequency equal to $f_{O2} = 239.729565$ Hz, the same frequency Henry observed when he detected beeps coming from Albert. Albert doesn't begin to receive these beeps until the he receives the last beep from Henry's trip to Gog, a time $t_1 = 20.0003480 \text{ y}$ from the start of the trip. Albert receives the beeps from Henry's return trip only during the remaining time, $t_2 = t - t_1 = 0.0003480 \text{ y}$, at the end of trip. This is just during the last 3 h of the trip. Henry is going so fast, that the beeps he is sending begin to arrive on Earth only 3 h before he arrives himself. During this time, Albert receives $f_{O2}t_2 = 2632812$ beeps. This is about 1 mo in Henry's time, which is how long the trip back to Earth took in Henry's frame of reference.

The total number of beeps received by Albert during his round trip is $2632812 + 2632812 = 5265624$ which is 2.0022910 mo in Henry's time, exactly equal to the total time of the trip in Henry's frame of reference.